About Lepton Flavor Violating Kaon Decays

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Strong experimental limits exist only for lepton flavor violating (LFV) decays in the mostly CP- odd K_L system, and in the K^+ system. A model independent analysis constrains $Br(K_S \to \pi^0 e\mu)$ to be below 7.2×10^{-12}, and in a large class of models, $Br(K_S \to e\mu)$ must be below 7.6×10^{-12}. Scenarios in which LFV decays occur at a high rate outside this class of models are qualitatively unlikely. Limits on $Br(K^+ \to \pi^+ e\mu)$ however can not be set without arguably blithe assumptions.

Consider first the 3 body decays. For $K^- \to \pi^- e\mu$, one naturally turns to the branching ratio limits from Brookhaven experiment[1] 865:

$$Br(K^+ \to \pi^+ e^- \mu^+) < 2.8 \times 10^{-11}$$
$$Br(K^+ \to \pi^+ e^+ \mu^-) < 5.2 \times 10^{-10}.$$

To apply these limits to the corresponding K^- processes, one needs to apply C, the charge conjugation operator. C is not a good symmetry in the known flavor physics of charged weak currents, and it is reasonable to ask if it may not be conserved in new flavor physics as well.

However, in the case where generation number is conserved, one may infer something from the combination of the above results and the KTeV[2] result (valid for either charge combination) $Br(K_L \to \pi^0 e\mu) < 3.3 \times 10^{-10}$. There is information about both the s and $\bar{s}$ components of the decay in the K_L case, and this may be combined with the information from the $\bar{s}$ components of the K^+ decay. With the conventions $CP\left|K^0\right\rangle = -\left|\overline{K}^0\right\rangle$ and $CP\left|\overline{K}^0\right\rangle = -\left|K^0\right\rangle$,

$$\Gamma\left(K_L \to \pi^0 e\mu\right) \propto \left| \frac{(1+\varepsilon)\left\langle \pi^0 e\mu \middle| K^0 \right\rangle + (1-\varepsilon)\left\langle \pi^0 e\mu \middle| \overline{K}^0 \right\rangle}{\sqrt{2}\sqrt{1+|\varepsilon|^2}} \right|^2 .$$

The equation holds for either $e^-\mu^+$ or $e^+\mu^-$ separately, but it does assume that there will be interference between the K^0 and $\overline{K}^0$ amplitudes. That may not be the case, and we will return to the non-interfering scenerio momentarily.

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[1] R.Appel *et.al.*, *Phys. Rev. Lett.*, **85** (2000) 2450 and **85** (2000) 2877

[2] A.Bellavance for the KTeV Collaboration, "Lepton Number Violating Processes at KTeV", DPF2002, May 24-28, Williamsburg, Virginia.

Approximate $\langle \pi^0 e\mu | K^0 \rangle = \langle \pi^+ e\mu | K^+ \rangle / \sqrt{2}$ and $\langle \pi^0 e\mu | \overline{K}^0 \rangle = \langle \pi^- e\mu | K^- \rangle / \sqrt{2}$ - the factor $\sqrt{2}$ is for the projection of a $d\bar{d}$ pair into a π^0. Then,

$$\Gamma(K_L) = \frac{|1-\varepsilon|^2}{4\left(1+|\varepsilon|^2\right)} \left| e^{i\chi} \frac{(1+\varepsilon)}{(1-\varepsilon)} \sqrt{\Gamma(K^+)} + \sqrt{\Gamma(K^-)} \right|^2$$

with χ the relative phases between the $\langle \pi^- e\mu | K^- \rangle$ and $\langle \pi^+ e\mu | K^+ \rangle$ amplitudes. For any given values of $\Gamma(K_L)$ and $\Gamma(K^+)$, the largest possible magnitude of $\Gamma(K^-)$ is when the K^+ and K^- terms destructively interfere. In that case,

$$\Gamma(K^- \to \pi^- e\mu) = \left(\sqrt{\frac{4\left(1+|\varepsilon|^2\right)\Gamma(K_L \to \pi^0 e\mu)}{|1-\varepsilon|^2}} + \frac{|1+\varepsilon|}{|1-\varepsilon|} \sqrt{\Gamma(K^+ \to \pi^+ e\mu)} \right)^2 .$$

The value of $\Gamma(K^- \to \pi^- e\mu)$ for any specific experimental values $\Gamma(K^+ \to \pi^+ e\mu)$ and $\Gamma(K_L \to \pi^0 e\mu)$ may thus be calculated.

All of this assumes that the lepton flavor violating decay is like, say, the two pion decays of the neutral kaon. Such need not be the case. If generation number is conserved, the charge of the leptons in the final states reveals if a strange or an antistrange quark decayed. Such is the case in $K_{\mu 3}$ of the K_L; in this case there is no interference as the trajectories of the system are distinguishable. In that case, we have

$$\Gamma(K_L \to \pi^0 e\mu) \propto \left| \frac{(1+\varepsilon)\langle \pi^0 e\mu | K^0 \rangle}{\sqrt{2}\sqrt{1+|\varepsilon|^2}} \right|^2 + \left| \frac{(1-\varepsilon)\langle \pi^0 e\mu | \overline{K}^0 \rangle}{\sqrt{2}\sqrt{1+|\varepsilon|^2}} \right|^2$$

and

$$\Gamma(K^- \to \pi^- e\mu) = \frac{4\left(1+|\varepsilon|^2\right)\Gamma(K_L \to \pi^0 e\mu)}{|1-\varepsilon|^2} - \frac{|1+\varepsilon|^2}{|1-\varepsilon|^2} \Gamma(K^+ \to \pi^+ e\mu).$$

For large values of $\Gamma(K^+ \to \pi^+ e\mu)$ and small values of $\Gamma(K_L \to \pi^0 e\mu)$ an unphysical negative partial width is obtained, saying that there are some locations within the experimentally allowed parameter space that are inconsistent with the conservation of generation number.

Figure 1 shows the limit on $\Gamma(K^- \to \pi^- e^+ \mu^-)$ for the case where there is interference. The maximum partial widths and branching ratios are:

With interference, $e^+\mu^-$:	$\Gamma = 8.79 \times 10^{-23}$ *MeV*	$Br = 1.65 \times 10^{-9}$
With interference, $e^-\mu^+$:	$\Gamma = 2.76 \times 10^{-23}$ *MeV*	$Br = 5.16 \times 10^{-10}$
Without interference:	$\Gamma = 1.68 \times 10^{-23}$ *MeV*	$Br = 3.17 \times 10^{-10}$

The results are the same regardless of lepton charge pairing for the non-interference case because the maximum happens when all of the K_L width is attributed to the s-quark K^- equivalent amplitude and the assumed K^+ width is set to zero.

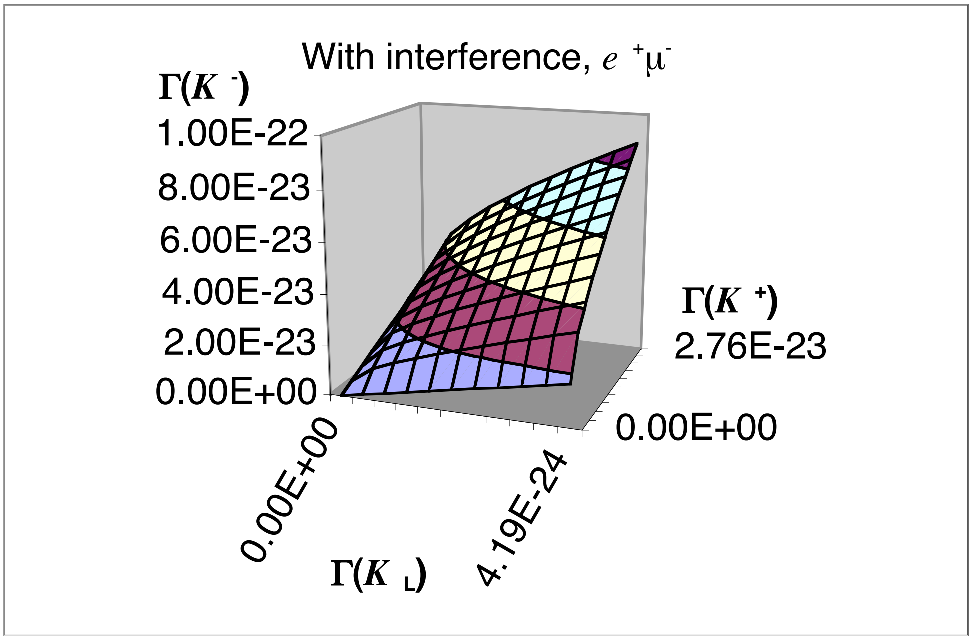

Figure 1

Now consider the 3 body LFV decays of K_S. Start again at

$$\Gamma\left(K_L \to \pi^0 e\mu\right) \propto \left| \frac{(1+\varepsilon)\left\langle \pi^0 e\mu \middle| K^0 \right\rangle + (1-\varepsilon)\left\langle \pi^0 e\mu \middle| \bar{K}^0 \right\rangle}{\sqrt{2}\sqrt{1+|\varepsilon|^2}} \right|^2 .$$

but now write $\left\langle \pi^0 e\mu \middle| \bar{K}^0 \right\rangle = \Psi_3 \left\langle \pi^+ e\mu \middle| K^+ \right\rangle / \sqrt{2}$ - the complex number Ψ_3 parametarizes how the ΔS=+*1* decay differs from the ΔS=-*1* decay. Then

$$\frac{\Gamma\left(K_L \to \pi^0 e\mu\right)}{\Gamma\left(K^+ \to \pi^+ e\mu\right)} = \frac{|1+\varepsilon|^2 + 2\Re\left[(1+\varepsilon)^*(1-\varepsilon)\Psi_3\right] + \left|\Psi_3(1-\varepsilon)\right|^2}{4\left(1+|\varepsilon|^2\right)},$$

where the electron-muon charge combination is the same for both K_L and K^+ modes. For any given pair of branching ratios for the K_L and K^+ semileptonic LFV decays, this constrains $\Psi_3 = \Psi_r + i\Psi_i$ into a circle on the complex plane:

$$\frac{4\left(1+|\varepsilon|^2\right)}{|1-\varepsilon|^2}\frac{Br\left(K_L \to \pi^0 e\mu\right)}{Br\left(K^+ \to \pi^+ e\mu\right)}\frac{\tau^+}{\tau_L} = \Psi_r^2 + \Psi_i^2 + \frac{2\left(1-|\varepsilon|^2\right)}{|1-\varepsilon|^2}\Psi_r + \frac{4\Im(\varepsilon)}{|1-\varepsilon|^2}\Psi_i + \frac{|1+\varepsilon|^2}{|1-\varepsilon|^2}.$$

The center of the circle only depends on the indirect mixing parameter ε. It is at $\Psi_r^{(0)} = -(1-|\varepsilon|^2)/|1-\varepsilon|^2 = -1.003318$ and $\Psi_i^{(0)} = -2\Im(\varepsilon)/|1-\varepsilon|^2 = -0.003145$. The radius does depend on the K_L and K^+ branching ratios however; it is

$$\begin{aligned} R^2 &= \frac{4\left(1+|\varepsilon|^2\right)}{|1-\varepsilon|^2}\frac{Br\left(K_L \to \pi^0 e\mu\right)}{Br\left(K^+ \to \pi^+ e\mu\right)}\frac{\tau^+}{\tau_L} - \frac{|1+\varepsilon|^2}{|1-\varepsilon|^2} + \frac{1}{4}\left(\frac{2\left(1-|\varepsilon|^2\right)}{|1-\varepsilon|^2}\right)^2 + \frac{1}{4}\left(\frac{4\Im(\varepsilon)}{|1-\varepsilon|^2}\right)^2 \\ &= 0.960\frac{Br\left(K_L \to \pi^0 e\mu\right)}{Br\left(K^+ \to \pi^+ e\mu\right)} \end{aligned}$$

where the K^+ lifetime has been taken as 12.384ns, and the K_L lifetime as 51.8ns. The cancellation of the last three terms is exact, and the radius of the circle, using the $Br(K^+ \to \pi^+ e^+ \mu^-)$ data from E865 and the KTeV result, is 0.780.

Now it is possible to constrain $Br(K_S \to \pi^0 e\mu)$. For each assumed value of $Br(K_L \to \pi^0 e\mu)$ and $Br(K^+ \to \pi^+ e\mu)$, the circular locus of Ψ_3 may be determined; for each point on that locus, $Br(K_S \to \pi^0 e\mu)$ may be determined using

$$\frac{Br\left(K_S \to \pi^0 e\mu\right)}{Br\left(K^+ \to \pi^+ e\mu\right)}\frac{\tau^+}{\tau_S} = \frac{|1+\varepsilon|^2 - 2\Re\left[(1+\varepsilon)^*(1-\varepsilon)\Psi_3\right] + |\Psi_3(1-\varepsilon)|^2}{4\left(1+|\varepsilon|^2\right)}.$$

This is the equation of a second circle with radius that goes as $\sqrt{Br\left(K_S \to \pi^0 e\mu\right)}$, centered at $(1.003318, 0.003145)$. Figure 1 shows the locus of Ψ_3. We do not know where on this locus Ψ_3 actually resides, but we know that for every point in the locus, the distance to point C_2 is proportional to $\sqrt{Br\left(K_S \to \pi^0 e\mu\right)}$. That distance is maximized when $\theta = \pi + atan(0.003145 / 1.003318)$.

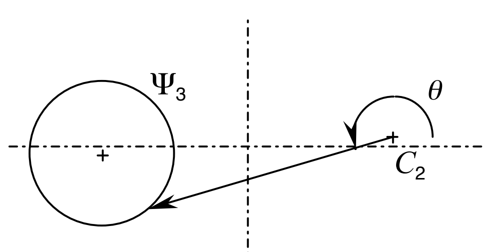

Figure 1

At that angle, the length of the arrow is the radius of the locus of Ψ_3 plus twice the distance from the origin to C_2:

$$\sqrt{\frac{4\left(1+|\varepsilon|^2\right)}{|1-\varepsilon|^2}\frac{Br\left(K_S\to\pi^0 e\mu\right)}{Br\left(K^+\to\pi^+ e\mu\right)}\frac{\tau^+}{\tau_S}}=\sqrt{\frac{4\left(1+|\varepsilon|^2\right)}{|1-\varepsilon|^2}\frac{Br\left(K_L\to\pi^0 e\mu\right)}{Br\left(K^+\to\pi^+ e\mu\right)}\frac{\tau^+}{\tau_L}}+2\sqrt{\left(\frac{\left(1-|\varepsilon|^2\right)}{|1-\varepsilon|^2}\right)^2+\left(\frac{2\Im(\varepsilon)}{|1-\varepsilon|^2}\right)^2}.$$

Whence

$$Br\left(K_S\to\pi^0 e\mu\right)=\left(\sqrt{Br\left(K_L\to\pi^0 e\mu\right)\frac{\tau_S}{\tau_L}}+\sqrt{\frac{Br\left(K^+\to\pi^+ e\mu\right)}{\left(1+|\varepsilon|^2\right)|1-\varepsilon|^2}\frac{\tau_S}{\tau^+}\left\{\left(1-|\varepsilon|^2\right)^2+\left(2\Im(\varepsilon)\right)^2\right\}}\right)^2.$$

The maximum possible value for $Br(K_S\to\pi^0 e\mu)$ is 7.24×10^{-12}, corresponding to a partial width of 5.32×10^{-23} MeV.

Finally, consider the two body decay $K_S\to e\mu$. While there is a sharp limit on $Br(K_L\to e\mu) < 4.7\times10^{-12}$, from BNL E871[3], there is no observational limit for the branching ratio; the best we can do is note that the 1σ uncertainty on the two dominant $\pi\pi$ branching ratios[4] is *0.14*%. The branching ratio is

$$\begin{aligned}Br\left(K_S\to e\mu\right)&=\frac{\Gamma\left(K_S\to e\mu\right)}{\Gamma\left(K_L\to e\mu\right)}\frac{\tau_S}{\tau_L}Br\left(K_L\to e\mu\right)\\&=\frac{\left|(1+\varepsilon)\left\langle e\mu\middle|K^0\right\rangle-(1-\varepsilon)\left\langle e\mu\middle|\bar{K}^0\right\rangle\right|^2}{\left|(1+\varepsilon)\left\langle e\mu\middle|K^0\right\rangle+(1-\varepsilon)\left\langle e\mu\middle|\bar{K}^0\right\rangle\right|^2}\frac{\tau_S}{\tau_L}Br\left(K_L\to e\mu\right).\end{aligned}$$

There is a pole where $\Psi_2 = \left\langle e\mu\middle|\bar{K}^0\right\rangle / \left\langle e\mu\middle|K^0\right\rangle$ is $(\varepsilon + 1) / (\varepsilon - 1)$, and it permits permits $K_S\to e\mu$ to happen rapidly, even in the presence of strong limits on $Br(K_L\to e\mu)$. Although it would be most unusual to find short range LFV physics coinciding with the physics of $K^0-\bar{K}^0$ mixing, which is very heavily influenced by long range QCD effects, the possibility can not be *a priori* ruled out. Without

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[3] D. Ambose *et.al.*, Phys. Rev. Lett. **81** (1998) 5734.
[4] S. Eidelman *et.al.*, Phys. Lett. **B592** (2004) 1.

further assumptions about what is the possible range of Ψ_2, we can not assign a probability to the likelyhood that Ψ_2 has a value corresponding to a certain $Br(K_S \to e\mu)$, but qualitatively it does seem unlikely.

To constrain $Br(K_S \to e\mu)$ is to constrain Ψ_2. It could be different for the two different electron-muon charge combinations, as generation number may be a conserved quantity.

$$\Psi_2(e^+\mu^-) = \langle e^+\mu^- | \overline{K}^0 \rangle / \langle e^+\mu^- | K^0 \rangle \qquad \Psi_2(e^-\mu^+) = \langle e^-\mu^+ | \overline{K}^0 \rangle / \langle e^-\mu^+ | K^0 \rangle$$

With the conventional generation number assignments, $\Psi_2(e^+\mu^-)$ is infinite and $\Psi_2(e^-\mu^+)$ is zero; with other generation number assignments, the opposite situation can arise.

Invariance under our most reliable symmetry, CPT, does not by itself constrain Ψ_2. Reinserting $\vec{p}_i$ and s_i for the momentum and helicity of the leptons,

$$\begin{aligned} CPT\langle e^+(\vec{p}_e,\lambda_e)\mu^-(\vec{p}_\mu,\lambda_\mu) | \overline{K}^0(\vec{p}_e+\vec{p}_\mu,\lambda_e-\lambda_\mu=0)\rangle &= CP\langle \overline{K}^0 | e^+(-\vec{p}_e,\lambda_e)\mu^-(-\vec{p}_\mu,\lambda_\mu)\rangle \\ &= -C\langle \overline{K}^0 | e^+(\vec{p}_e,-\lambda_e)\mu^-(\vec{p}_\mu,-\lambda_\mu)\rangle \\ &= -\langle K^0 | e^-(\vec{p}_e,-\lambda_e)\mu^+(\vec{p}_\mu,-\lambda_\mu)\rangle \\ &= -\langle e^-(\vec{p}_e,-\lambda_e)\mu^+(\vec{p}_\mu,-\lambda_\mu) | K^0\rangle^*, \end{aligned}$$

so $\Psi_2(e^+\mu^-) = -\langle e^-(\vec{p}_e,-\lambda_e)\mu^+(\vec{p}_\mu,-\lambda_\mu) | K^0\rangle^* / \langle e^+(\vec{p}_e,\lambda_e)\mu^-(\vec{p}_\mu,\lambda_\mu) | K^0\rangle$. If the amplitude changes sign when the charges and helicities of the outgoing particles are simultaneously flipped, the magnitude of Ψ_2 will be one. We may set Ψ_2 equal to one if we assume invariance under the application of C to the initial but not final state, or if we assume various other invariances in conjunction with P. However, no particularly simple symmetry principle constrains Ψ_2.

In a large class of models, it is possible to constrain $K_S \to e\mu$ with the above limit for $K_S \to \pi^0 e\mu$. The scheme is to take the three body decay limit and interpret it as a limit on the couplings in an effective theory that allows for $s(\bar{s}) \to d(\bar{d})e\mu$ transitions, where the $s(\bar{s})$ combination is the same as in a K_S meson. Those couplings can then be used to constrain the two body decay.

As long as the new physics is at a large mass scale, the sum of all the possible diagrams permitted by the new physics is the sum of five diagrams of the form shown in figure 2, where the lepton couplings Γ are $\{1, \gamma^5, \gamma^\mu, \gamma^5\gamma^\mu, \sigma^{\mu\nu}\}$. The hadronic currents must be constructed from two 4-momenta, p_K and p_π; this permits scalar, vector, and tensor hadronic currents.

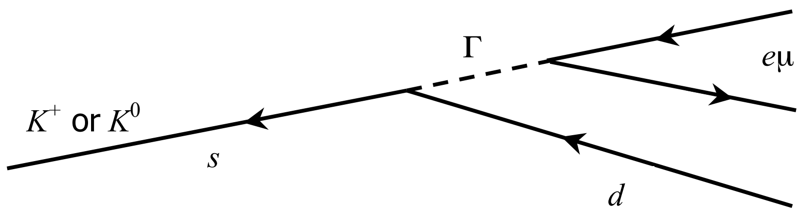

Figure 2

The matrix element for figure 2 may be written as

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\begin{aligned}
M_3 = &\langle \pi | \bar{s} d | K \rangle \quad \left[\bar{u}(p_\mu)\left(a_S + a_P \gamma^5\right) v(p_e) \right] \\
&+ \langle \pi | \bar{s} \gamma^\alpha d | K \rangle \left[\bar{u}(p_\mu)\left(a_V + a_A \gamma^5\right) \gamma_\alpha v(p_e) \right] \\
&+ \langle \pi | \bar{s} \sigma^{\alpha\beta} d | K \rangle \left[\bar{u}(p_\mu)\left(a_T\right) \sigma_{\alpha\beta} v(p_e) \right]
\end{aligned}
$$

with a_i the coupling constants of the new physics. These coupling constants have the same dimension as G_F. For $s \to de\mu$, a similar matrix element is relevant, but with potentially different couplings $\bar{a}_i$.

For $K_S \to \pi^0 e\mu$, the decays of the s and $\bar{s}$ components will not interfere if there is a conserved generation number; in that case one can distinguish an s decay from an $\bar{s}$ decay by observing the charge combination of the leptons. The partial width computed using couplings $(1+\varepsilon)a_i/\sqrt{2\left(1+|\varepsilon|^2\right)}$ would be added to the partial width computed using $(1-\varepsilon)\bar{a}_i/\sqrt{2\left(1+|\varepsilon|^2\right)}$. Or, it may happen that the amplitudes do interfere. In either case, the combination of a_i and $\bar{a}_i$ that is valid for the three body decay is the same combination needed for the two body decay. All that is needed is to define new couplings $a_i^{(S)}$ and treat them in the same manner as the old couplings a_i. Having done that, the (S) superscript becomes redundant and I drop it.

The K_S decay does differ from the quark-level one in that for the neutral kaons, an additional factor of $1/\sqrt{2}$ for the projection of a $d\bar{d}$ pair into a π^0 is needed that does not appear in the quark level, or for that matter in the charged kaon, decay.

The tensor hadronic current term, typically written as

$$\langle\pi|\bar{s}\sigma^{\alpha\beta}d|K\rangle\left[\bar{u}(p_\mu)\sigma_{\alpha\beta}\,v(p_e)\right]=c_2(2f_T/m_K)[p_K]^\alpha[p_\pi]^\beta\left[\bar{u}(p_\mu)\sigma_{\alpha\beta}\,v(p_e)\right]$$
$$=\frac{c_2 f_T}{m_K}[p_K-p_\pi]^\alpha[p_K+p_\pi]^\beta\left[\bar{u}(p_\mu)\sigma_{\alpha\beta}\,v(p_e)\right],$$

will not contribute to the two body decay, as there is only one 4-momentum to construct a hadronic current from. The leptonic current $\bar{u}_\mu\sigma_{\alpha\beta}v_e$ is antisymmetric under exchange of the indices α and β, the hadronic current must be also antisymmetric in order for there to be a net contribution. But there is no way to construct an antisymmetric combination out of a single 4-vector. If we had another 4-vector s^θ to work with, along with the kaon 4-momentum p^ϕ, then we could write a hadronic current proportional to $\varepsilon_{\alpha\beta\theta\phi}\, s^\theta\, p^\phi$ but such is not the case.

With this in mind, we can only proceed under the assumption that the tensor terms are negligible; otherwise it is possible for a large tensor amplitude to cancel the scalar and vector amplitudes in the three body decay and not in the two body decay. This is the restriction to a category of models alluded to above. An example of such models is horizontal gauge symmetry[5], in which the new LFV physics is produced by a single boson of either spin zero or spin one.

The vector hadronic current term may be taken from $K_{\ell 3}$ measurements

$$\langle\pi|\bar{s}\gamma^\alpha d|K\rangle=\left(f_+[p_K+p_\pi]^\alpha+f_-[p_K-p_\pi]^\alpha\right)=f_+\left([p_K+p_\pi]^\alpha+\xi[p_K-p_\pi]^\alpha\right).$$

A certain imprecision is implicit in the notation; $\langle\pi|\bar{s}\gamma^\alpha d|K\rangle$ denotes currents that annihilate either s, $\bar{s}$, or any linear combination thereof. If there is in fact a difference in the coupling constants at the quark-boson vertex for Figure 2 or its antimatter conjugate, I define that difference to reside in the redefined coupling constants a_i.

I use

$$f_\pm=f_\pm(0)\left[1+\lambda_\pm\left(\frac{(p_K-p_\pi)^2}{m_\pi^2}\right)\right]$$

with $f_+(0)=1$, $\lambda_+=0.030$, $f_-(0)=-0.0025$, and $\lambda_-=0.0$. These numbers are broadly consistent with results available before 2004 in both the K_L and K^+ and are more than adequately precise for the analysis at hand. In any case, use of the more precise results of Andre[6] and KTeV[7] are unjustified in the absence of a careful

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[5] R.H.Cahn and H.Harari, *Nucl.Phys.*, **B176** (1980) 135
[6] T. Andre, “Radiative Corrections in K^0_{l3} Decays”, hep-ph/0406006
[7] T.Alexopolulos *et.al.* “Measurements of Semileptonic K_L Decay Form-factors”, hep-ex/0406003

analysis of radiative corrections for the $e\mu$ mode. Consistent with the data, ξ is taken to be real.

The scalar hadronic current $\langle\pi|\bar{s}d|K\rangle$ is determined from the vector current through the relation

$$\langle\pi|\bar{s}d|K\rangle = \frac{[p_\pi - p_K]_\alpha \langle\pi|\bar{s}\gamma^\alpha d|K\rangle}{m_s - m_d}.$$

The relevant matrix element for the three body decay $K_S \to \pi^0 e\mu$ is therefore

$$M_3 = f_+\left([p_K + p_\pi]^\alpha + \xi[p_K - p_\pi]^\alpha\right)$$
$$\left\{\frac{[p_\pi - p_K]_\alpha}{m_s - m_d}\left[\bar{u}(p_\mu)(a_S + a_P\gamma^5)v(p_e)\right] + \left[\bar{u}(p_\mu)(a_V + a_A\gamma^5)\gamma_\alpha v(p_e)\right]\right\}.$$

Evaluation of $|M_3|^2$ is straightforward. Some of the relations that occur in the intermediate steps are listed in the appendix. There are three terms: one corresponding to a *J=0* boson exchange, one corresponding to a *J=1* boson exchange, and an interference term. They are, respectively,

$$\overline{|M_3(J=0)|^2} = 4\,\frac{|a_S|^2 + |a_P|^2}{(m_s - m_d)^2}\left(H_3\cdot[p_K - p_\pi]\right)^2 (p_e\cdot p_\mu),$$

$$\overline{|M_3(J=1)|^2} = 4\left(|a_V|^2 + |a_P|^2\right)\left[2(H_3\cdot p_\mu)(H_3\cdot p_e) - H_3^\alpha(p_e\cdot p_\mu)\right],$$

and

$$\overline{|M_3(J=\otimes)|^2} = 8m_\mu\,\frac{\Re\left(a_A a_P^* - a_V a_S^*\right)}{(m_s - m_d)}\,H_3\cdot[p_K - p_\pi](H_3\cdot p_e),$$

where the hadronic current $H_3^\alpha = f_+\left([p_K + p_\pi]^\alpha + \xi[p_K - p_\pi]^\alpha\right)$.

The partial widths as found by integrating over phase space with the RAMBO algorithm are

$$\Gamma_3(J=0) = \frac{1}{2m_k}\int d\Phi\,\overline{|M_3(J=0)|^2} = \left(8.56\times10^{13}\,MeV^7\right)\frac{|a_S|^2 + |a_P|^2}{(m_s - m_d)^2},$$
$$\Gamma_3(J=1) = \frac{1}{2m_k}\int d\Phi\,\overline{|M_3(J=1)|^2} = \left(5.74\times10^{8}\,MeV^5\right)\left(|a_V|^2 + |a_A|^2\right)$$

and

$$\Gamma_3(J=\otimes)=\frac{1}{2m_k}\int d\Phi\,\overline{\left|M_3(J=\otimes)\right|^2}=\left(2.51\times10^{11}\,MeV^6\right)\frac{\Re\left(a_A a_P^* - a_V a_S^*\right)}{\left(m_s-m_d\right)}.$$

The sum of these three must be less than $2\Gamma(K_S\to\pi^0 e\mu)$ = 10.6 x10^{-23} MeV, where the factor of 2 is equivalent to the factor of $\sqrt{2}$ in the couplings needed to allow for the projection of a $d\bar{d}$ pair into a π^0.

The matrix element for $K_S \to e\mu$ is

$$\begin{aligned} M_2 = &\langle 0|\bar{s}\gamma^5 d|K\rangle \quad \left[\bar{u}(p_\mu)\left(a_S+a_P\gamma^5\right)v(p_e)\right] \\ &+\langle 0|\bar{s}\gamma^5\gamma^\alpha d|K\rangle \quad \left[\bar{u}(p_\mu)\left(a_V+a_A\gamma^5\right)\gamma_\alpha v(p_e)\right]. \end{aligned}$$

The axial-vector current is taken from measurement of $K^+ \to \mu^+\nu$ and $u\leftrightarrow d$ isospin:

$$\begin{aligned} \langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|K^+(p_K)\right\rangle = if_K\left[p_K\right]^\alpha \quad &\Rightarrow \quad \langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|K^0(p_K)\right\rangle = if_K\left[p_K\right]^\alpha \\ &\Rightarrow \quad \langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|\bar{K}^0(p_K)\right\rangle = if_K\left[p_K\right]^\alpha, \end{aligned}$$

Since $CP\left|K^0\right\rangle = -\left|\bar{K}^0\right\rangle$ and $CP(\bar{s}\gamma^5\gamma^\alpha d)CP^{-1} = -(\bar{d}\gamma^5\gamma_\alpha s)$. Then

$$\begin{aligned} \langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|K_S(p_K)\right\rangle &= \frac{1+\varepsilon}{\sqrt{2}\sqrt{1+|\varepsilon|^2}}\langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|K^0(p_K)\right\rangle - \frac{1-\varepsilon}{\sqrt{2}\sqrt{1+|\varepsilon|^2}}\langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|\bar{K}^0(p_K)\right\rangle \\ &= \frac{\sqrt{2}\varepsilon}{\sqrt{1+|\varepsilon|^2}} if_K\left[p_K\right]^\alpha \end{aligned}$$

where f_K = 160 MeV. Similarly,

$$\langle 0|\bar{s}\gamma^5\gamma^\alpha d\left|K_L(p_K)\right\rangle = \frac{\sqrt{2}}{\sqrt{1+|\varepsilon|^2}} if_K\left[p_K\right]^\alpha,$$

showing that the partial width of $K_S \to e\mu$ is supressed relative to $K_L \to e\mu$ by a factor of $|\varepsilon|^2$.

The pseudoscalar hadronic current $\langle\pi|\bar{s}\gamma^5 d|K\rangle$ is determined from the axial vector current through the relation

$$\langle 0|\bar{s}\gamma^5 d|K\rangle = \frac{\left[p_K\right]_\alpha\langle 0|\bar{s}\gamma^5\gamma^\alpha d|K\rangle}{m_s+m_d}.$$

The relevant matrix element for the two body decay $K_S \to e\mu$ is therefore

$$M_2 = \frac{\sqrt{2}\varepsilon i}{\sqrt{1+|\varepsilon|^2}} f_K [p_K]^\alpha \left\{ \frac{[p_K]_\alpha}{m_s + m_d} \left[\bar{u}(p_\mu)(a_S + a_P\gamma^5)v(p_e)\right] + \left[\bar{u}(p_\mu)(a_V + a_A\gamma^5)\gamma_\alpha v(p_e)\right] \right\}.$$

Evaluation of $|M_2|^2$ is also straightforward, and can be done by inspection of the $|M_3|^2$ result:

$$\overline{|M_2(J=0)|^2} = 4\,\frac{|a_S|^2 + |a_P|^2}{(m_s + m_d)^2}\,(H_2 \cdot p_K)^2\,(p_e \cdot p_\mu),$$

$$\overline{|M_2(J=1)|^2} = 4\,\left(|a_V|^2 + |a_P|^2\right)\left[2(H_2 \cdot p_\mu)(H_2 \cdot p_e) - H_2^2(p_e \cdot p_\mu)\right],$$

and

$$\overline{|M_2(J=\otimes)|^2} = -8m_\mu\,\frac{\Re\left(a_A a_P^* - a_V a_S^*\right)}{(m_s + m_d)}\,(H_2 \cdot p_K)\,(H_2 \cdot p_e),$$

where the hadronic current $H_2^\alpha = \left|\sqrt{2}\varepsilon i \big/ \sqrt{1+|\varepsilon|^2}\right| f_K\ p_K^\alpha$, has an unobservable phase removed.

The partial widths are

$$\Gamma_2(J=0) = \frac{1}{2m_k}\int d\Phi\,\overline{|M_2(J=0)|^2} = \left(2.95\times 10^{11}\ MeV^7\right)\frac{|a_S|^2 + |a_P|^2}{(m_s + m_d)^2},$$

$$\Gamma_2(J=1) = \frac{1}{2m_k}\int d\Phi\,\overline{|M_2(J=1)|^2} = \left(5.37\times 10^{4}\ MeV^5\right)\left(|a_V|^2 + |a_A|^2\right)$$

and

$$\Gamma_2(J=\otimes) = \frac{1}{2m_k}\int d\Phi\,\overline{|M_2(J=\otimes)|^2} = \left(-2.52\times 10^{8}\ MeV^6\right)\frac{\Re\left(a_A a_P^* - a_V a_S^*\right)}{(m_s + m_d)}.$$

The limit upon $\Gamma(K_S \to e\mu)$ is found with the following algorithm:

> Scan $|a_V|^2+|a_A|^2$ between *0* and *10.6* x10^{-23} / *5.74* x10^{8}; from the $\Gamma(K_S\to\pi^0 e\mu)$ constraint, that is the maximum possible range.
>
> > For each such value of $|a_V|^2+|a_A|^2$, scan $(|a_S|^2+|a_P|^2)$ / $(m_s - m_d)^2$

between 0 and $10.6 \times 10^{-23} / 8.56 \times 10^{13}$; again, the maximum value.

For each such combination of $(|a_S|^2+|a_P|^2) / (m_s - m_d)^2$ and $|a_V|^2+|a_A|^2$, compute the value of $\Re(a_A a_P^* - a_V a_S^*) / (m_s - m_d)$ so as to saturate the bound on $\Gamma(K_S \to \pi^0 e\mu)$

Using these values for $(|a_S|^2+|a_P|^2) / (m_s - m_d)^2$, $|a_V|^2+|a_A|^2$, and $\Re(a_A a_P^* - a_V a_S^*) / (m_s - m_d)$, and a reasonable estimate of $(m_s - m_d) / (m_s + m_d)$, write down the sum of the three terms given above for Γ_2. Report the largest such value.

The values of m_s and m_d to be used are current masses. As m_d is small relative to m_s, the ratio $(m_s - m_d) / (m_s + m_d)$ is reasonably well known. It probably lies between 0.89 and 0.91. For a value of 0.90, the above algorithm yields

$$\Gamma(K_S \to e\mu) < 4.02 \times 10^{-25} \text{ MeV} \qquad Br(K_S \to e\mu) < 7.56 \times 10^{-12}.$$

Variation of $(m_s - m_d) / (m_s + m_d)$ by ± 0.01 changes the result in the third digit, which is insignificant since the experimental input is given to only two significant digits.

Since the interference term enters with a positive sign in $\Gamma(K_S \to \pi^0 e\mu)$ and with a negative sign in $\Gamma(K_S \to e\mu)$, it is not surprising that the minimum value occurs when the interference term destructively interferes in $\Gamma(K_S \to \pi^0 e\mu)$. For the case where there are only $J=0$ couplings and neither interference nor $J=1$ couplings, $Br(K_S \to e\mu) < 5.57 \times 10^{-12}$. When there are only $J=1$ couplings, helicity suppression is evident: $Br(K_S \to e\mu) < 1.87 \times 10^{-13}$. The ratio $5.57 \times 10^{-12} / 1.87 \times 10^{-13}$ is close to, but not exactly, $(m_K / m_\mu)^2$. Were we comparing the branching ratios under the condition $|a_S|^2+|a_P|^2 = |a_V|^2+|a_A|^2$, the ratio would be exactly $(m_K / m_\mu)^2$, but we are comparing branching ratios under the condition that

$$|a_S|^2 + |a_P|^2 = \frac{\Gamma_{MAX}(K_S \to \pi e\mu)(m_s - m_d)^2}{\left(\int d\Phi\, \overline{|M_3(J=0)|^2}\right) / \frac{|a_S|^2 + |a_P|^2}{(m_s - m_d)^2}},\ |a_V|^2 + |a_A|^2 = \frac{\Gamma_{MAX}(K_S \to \pi e\mu)}{\left(\int d\Phi\, \overline{|M_3(J=1)|^2}\right) / \left(|a_S|^2 + |a_P|^2\right)}$$

In other words, helicity suppression has a slightly different effect in the three body decays used to determine the coupling constant than it has it the two body decays where the coupling constants are employed.

All of the limits on K_S branching ratios found in this note are far beyond the reach of present or planned experiments.

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Appendix

Intermediate terms for $|M_3|^2$: (electron mass neglected)

a)
$$\sum_{e,\mu\ spins} \overline{u}(p_\mu)\left(a_V + a_A\gamma^5\right)\gamma_\alpha v(p_e)\overline{v}(p_e)\gamma_\beta\left(a_V^* - a_A^*\gamma^5\right)u(p_\mu) =$$
$$4\left(\left|a_V\right|^2 + \left|a_A\right|^2\right)\left(\left[p_e\right]_\alpha\left[p_\mu\right]_\beta + \left[p_e\right]_\beta\left[p_\mu\right]_\alpha - g_{\alpha\beta}\left(p_e\cdot p_\mu\right)\right)$$

b)
$$\frac{\left[p_K - p_\pi\right]_\alpha\left[p_K - p_\pi\right]_\beta}{\left(m_s - m_d\right)^2}\sum_{e,\mu\ spins} \overline{u}(p_\mu)\left(a_S + a_P\gamma^5\right)v(p_e)\overline{v}(p_e)\left(a_S^* - a_P^*\gamma^5\right)u(p_\mu) =$$
$$4\frac{\left[p_K - p_\pi\right]_\alpha\left[p_K - p_\pi\right]_\beta}{\left(m_s - m_d\right)^2}\left(\left|a_S\right|^2 + \left|a_P\right|^2\right)\left(p_e\cdot p_\mu\right)$$

c)
$$\frac{-\left[p_K - p_\pi\right]_\beta}{m_s - m_d}\sum_{e,\mu\ spins} \overline{u}(p_\mu)\left(a_V + a_A\gamma^5\right)\gamma_\alpha v(p_e)\overline{v}(p_e)\left(a_S^* - a_P^*\gamma^5\right)u(p_\mu) =$$
$$4\frac{m_\mu\left[p_K - p_\pi\right]_\beta}{m_s - m_d}\left(a_A a_P^* - a_V a_S^*\right)\left[p_e\right]_\alpha$$

d)
$$\frac{-\left[p_K - p_\pi\right]_\alpha}{m_s - m_d}\sum_{e,\mu\ spins} \overline{u}(p_\mu)\left(a_S + a_P\gamma^5\right)v(p_e)\overline{v}(p_e)\gamma_\beta\left(a_V^* - a_A^*\gamma^5\right)u(p_\mu) =$$
$$4\frac{m_\mu\left[p_K - p_\pi\right]_\alpha}{m_s - m_d}\left(a_A a_P^* - a_V a_S^*\right)^*\left[p_e\right]_\beta$$